RESEARCH INTO ADVANCED CONCEPTS

OF MICROWAVE POWER AMPLIFICATION

AND

GENERATION UTILIZING LINEAR BEAM DEVICES

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ABSTRACT

This is an interim report which summarizes work during the past six months on a theoretical study of some aspects of the interaction between a drifting stream of electrons with transverse cyclotron motions and an electromagnetic field. Particular emphasis is given to the possible generation and amplification of millimeter waves. The report includes brief discussions of the waves of a spiraling filamentary electron beam and the characteristics of magnetically focused, solid electron beams, with relativistic effects included in both cases.

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I. INTRODUCTION

The objective of this research program is to explore theoretically some aspects of the interaction between a drifting stream of electrons having transverse cyclotron motions and an electromagnetic field; particular emphasis being given to the possible generation and amplification of millimeter waves. Because of the interest in the possible applications to millimeter wavelengths, this study concentrates on electron stream-electromagnetic field interactions which involve a uniform, or fast-wave, circuit structure.

This interim report summarizes work on two aspects of the interaction between electron beams and electromagnetic fields. Section II discusses the waves of a spiraling filamentary electron beam and their coupling to the TEM waves of a uniform circuit. A coupled mode analysis is employed, and relativistic effects are included. Section III is concerned with relativistic electron beams of finite cross section. First the final results for the d-c characteristics of an uniform charge density electron beam are presented. A study of the transverse waves of a Brillouin beam has been initiated, and the approach being taken is discussed briefly.

II. SPIRALING FILAMENTARY ELECTRON BEAM

A. Spiraling Electron Beam Devices

Many electron beam devices whose operation is based on cyclotron resonance utilize a spiraling or rotating electron beam to provide the d-c power for the gain or oscillation mechanism.[1-4] In these devices the rotational beam motion is a necessary condition for the interaction to occur. A study of the characteristic waves of a spiraling filamentary electron beam has been undertaken to explore the possible interaction mechanisms possible when this type of electron beam couples to a circuit. In this study only uniform circuits which support fast waves will be included, since the emphasis is on possible applications to millimeter wavelengths.

It is the purpose of this study to clarify the basic theory of possible interactions, to suggest possible device configurations, and perhaps to indicate some of the design parameter values of importance. The analytical approach used in this study will be coupled mode theory. This has been extensively exploited previously for the analysis of devices using straight filamentary electron beams,[5,6] particularly parametric amplifiers. The coupled mode theory for the waves on a filamentary electron beam has been generalized to apply to an electron beam whose d-c motion is a spiral, or corkscrew. Coupled mode theory is, of course, an approximate analysis which neglects several factors. However, it does provide a powerful tool to examine the basic interaction mechanisms in a clear and concise manner.

The discussion of the spiraling filamentary electron beam presented here will be an interim report. The waves which can exist on a spiraling beam will be presented, together with an indication of the conditions for possible interaction with a fast-wave circuit. However, the details of possible electron beam-circuit interaction have yet to be explored.

B. D-C Electron Beam Equations

The basic model for the unperturbed, or d-c, spiraling filamentary electron beam is shown in Figure 1. The electrons are assumed to move in a helical path with radius r_o and pitch $2\pi/\beta_o$, with the axis of the helix parallel to a uniform d-c magnetic flux density B_o in the z direction. The d-c current along the helix is I_o, and the d-c velocities in the axial and angular directions are, respectively, $\dot{z}_o$ and $\dot{\theta}_o = \beta_o \dot{z}_o$. The d-c position of an electron is then given by

$$x_o = r_o \cos (\beta_o \dot{z}_o t + \phi) \quad , \qquad (1a)$$

$$y_o = r_o \sin (\beta_o \dot{z}_o t + \phi) \quad , \tag{1b}$$
$$z_o = \dot{z}_o t \quad , \tag{1c}$$

where ϕ is a phase angle introduced for generality.

Relativistic equations of motion will be employed since relativistic effects can be of considerable importance in rotating electron beams. Assuming that only electric and magnetic forces need be considered, these equations of motion are

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\gamma^2}}\right) = -\frac{e}{m}(E_x + \dot{y}B_z - \dot{z}B_y) \quad , \tag{2a}$$

$$\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{1-\gamma^2}}\right) = -\frac{e}{m}(E_y + \dot{z}B_x - \dot{x}B_z) \quad , \tag{2b}$$

$$\frac{d}{dt}\left(\frac{\dot{z}}{\sqrt{1-\gamma^2}}\right) = -\frac{e}{m}(E_z + \dot{x}B_y - \dot{y}B_x) \quad . \tag{2c}$$

Here e is the magnitude of the electronic charge, m is the rest mass of the electron, and

$$\gamma^2 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2 \tag{3}$$

where c is the velocity of light.

Inserting the assumed d-c motion of the electron beam given in Equations (1) into the equations of motion (2),

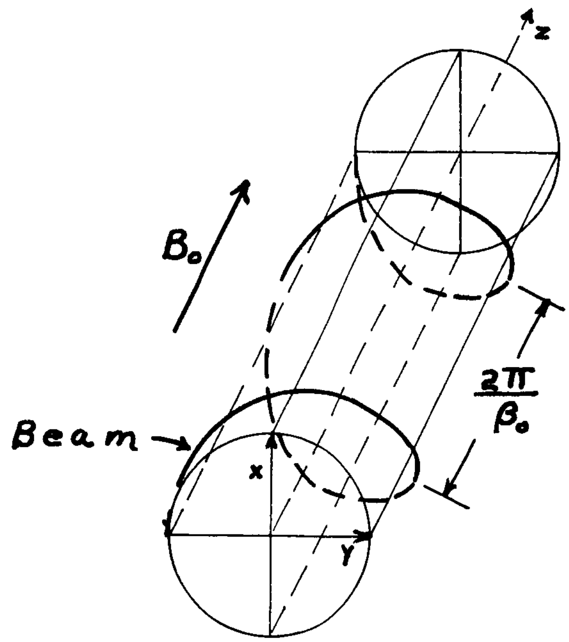

FIGURE 1. Model for Spiraling Filamentary Electron Beam.

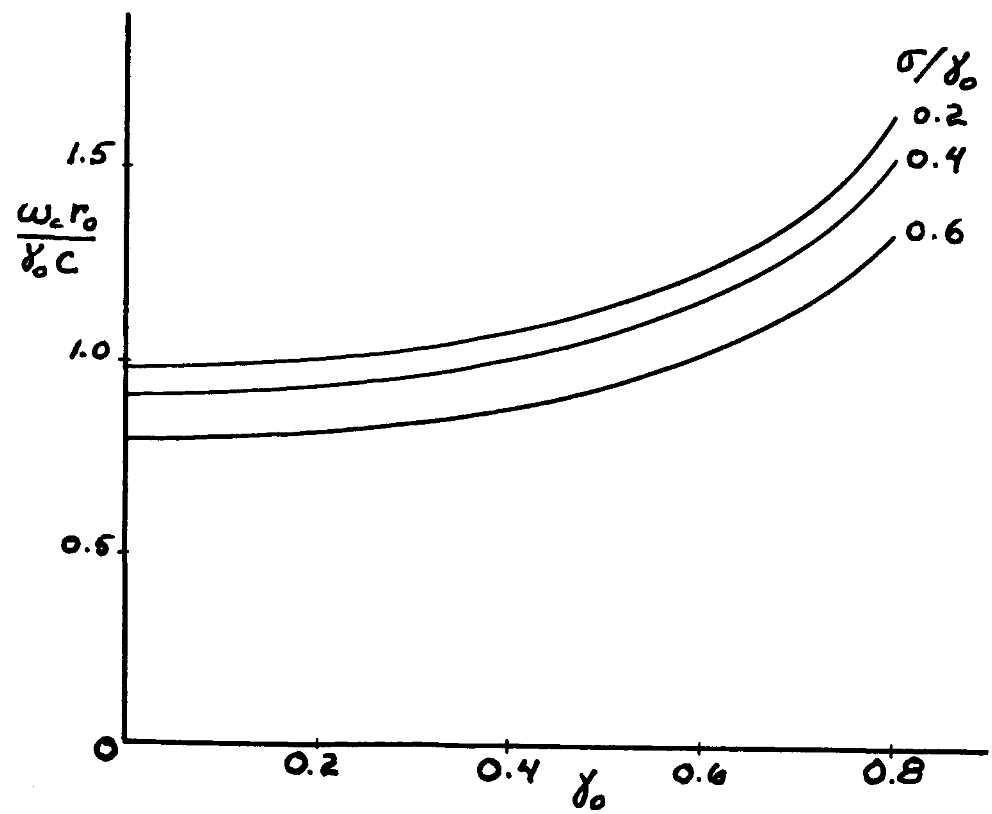

FIGURE 2. Normalized Radius $\omega_c r_o/\gamma_o c$ Versus γ_o, with σ/γ_o as Parameter.

the relationship between the angular and axial velocities is determined. The only electromagnetic field which enters into the d-c equations of motion is the axial magnetic flux density, B_o. Because a filamentary electron beam is assumed, the electric field due to space charge and the self-magnetic field of the beam current are neglected. With the following definitions

$$\eta = \sqrt{1 - \gamma_o^2} = \frac{1}{1 + \frac{e}{m}\frac{V_o}{c^2}} \quad , \tag{4a}$$

$$\omega_c = \frac{e}{m} B_o \quad , \tag{4b}$$

$$\sigma = \frac{\dot{z}_o}{c} \quad , \tag{4c}$$

where V_o is the d-c beam voltage, the d-c angular velocity of the electrons is

$$\theta_o = \sqrt{1 - \gamma_o^2}\ \omega_c = \eta\omega_c \quad . \tag{5}$$

For a given total d-c velocity γ_o, axial velocity σ, and cyclotron frequency ω_c, the radius of the spiral motion of the electrons is fixed;

$$\frac{\omega_c r_o}{\gamma_o c} = \sqrt{\frac{1 - (\frac{\sigma}{\gamma_o})^2}{1 - \gamma_o^2}} \quad . \tag{6}$$

Figure 2 shows this normalized spiral radius versus γ_o

for several values of the parameter σ/γ_o.

C. A-C Electron Beam Equations

A small-signal analysis is made in the usual way by setting $x = x_o + x_1$, $y_o + y_1$, $z = z_o + z_1$, where the a-c terms x_1, y_1, and z_1 are presumed to be small and proportional to exp $(j\omega t)$. To first order in the a-c quantities

$$\frac{\dot{x}}{\sqrt{1-\gamma^2}} \cong \frac{\dot{x}_1}{\eta} + \frac{\dot{x}_o(\dot{x}_o\dot{x}_1 + \dot{y}_o\dot{y}_1 + \dot{z}_o\dot{z}_1)}{c^2\ \eta^3} \quad , \tag{7}$$

with corresponding expressions for $\dot{y}/\sqrt{1-\gamma^2}$, and $\dot{z}/\sqrt{1-\gamma^2}$. Also to first order the total time derivative d/dt is equal to $\dot{z}_o(j\beta_e + \partial/\partial z)$ where, as usual, $\beta_e = \omega/\dot{z}_o$. The first order a-c equations of motion obtained from Equations (2) are

$$(j\beta_e + \frac{\partial}{\partial z})\left[\frac{\dot{x}_1}{\eta} + \frac{\dot{x}_o(\dot{x}_o\dot{x}_1+\dot{y}_o\dot{y}_1+\dot{z}_o\dot{z}_1)}{c^2\ \eta^3}\right] = \frac{-e}{m\dot{z}_o}(E_x+\dot{y}_oB_z-\dot{z}_oB_y)-\beta_c\dot{y}_1, \tag{8a}$$

$$(j\beta_e + \frac{\partial}{\partial z})\left[\frac{\dot{y}_1}{\eta} + \frac{\dot{y}_o(\dot{x}_o\dot{x}_1+\dot{y}_o\dot{y}_1+\dot{z}_o\dot{z}_1)}{c^2\ \eta^3}\right] = \frac{-e}{m\dot{z}_o}(E_y+\dot{z}_oB_x-\dot{x}_oB_z)+\beta_c\dot{x}_1, \tag{8b}$$

$$(j\beta_e + \frac{\partial}{\partial z})\left[\frac{\dot{z}_1}{\eta} + \frac{\dot{z}_o(\dot{x}_o\dot{x}_1+\dot{y}_o\dot{y}_1+\dot{z}_o\dot{z}_1)}{c^2\,\eta^3}\right] = \frac{-e}{m\dot{z}_o}\,(E_z+\dot{x}_oB_y-\dot{y}_oB_x), \tag{8c}$$

where $\beta_c = \omega_c/\dot{z}_o$. Also, note that $\beta_o = \dot{\theta}_o/\dot{z}_o = \eta\beta_c$.

As in the straight filamentary electron beam analysis, it is convenient to introduce circularly polarized components. Therefore we define

$$u_{\pm} = \dot{x}_1 \pm j\dot{y}_1 \quad , \tag{9a}$$

$$E_{\pm} = E_x \pm jE_y \quad , \tag{9b}$$

$$B_{\pm} = B_x \pm jB_y \quad , \tag{9c}$$

Because of the spiral character of the d-c electron beam, it turns out that in identifying the electron beam waves it is very convenient to introduce the phase factor $\psi = (\beta_o z + \phi) = (\eta\beta_c z + \phi)$, setting

$$u'_{\pm} = u_{\pm}\; e^{\mp j\psi} \tag{10a}$$

$$E'_{\pm} = E_{\pm}\; e^{\mp j\psi} \tag{10b}$$

$$B'_{\pm} = B_{\pm}\; e^{\mp j\psi} \quad . \tag{10c}$$

A parameter expressing the spiral character of the d-c electron beam is also introduced

$$\lambda = \frac{\left(\frac{\omega_c r_o}{c}\right)^2}{2 + \left(\frac{\omega_c r_o}{c}\right)^2} = \frac{1 - \eta^2 - \sigma^2}{1 + \eta^2 - \sigma^2} \tag{11}$$

Note that for a straight filamentary electron beam, $\lambda = 0$.

With the definitions given above, the first order a-c equations of motion (8) become

$$\left(\frac{\partial}{\partial z} + j\beta_e + j\lambda\eta\beta_c\right)u'_+ - \lambda\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_e\right)u'_- + j\,\frac{2\,\sigma\sqrt{\lambda}}{\sqrt{1+\eta^2-\sigma^2}}\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_c\right)\dot{z}_1$$

$$= \frac{-2\,\eta^3 e}{m(1+\eta^2-\sigma^2)}\left(\frac{E'_+}{\dot{z}_o} + jB'_+ + \eta\beta_c r_o B_z\right) \quad , \qquad (12a)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e - j\lambda\eta\beta_c\right)u'_- - \lambda\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)u'_+ - j\,\frac{2\,\sigma\sqrt{\lambda}}{\sqrt{1+\eta^2-\sigma^2}}\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)\dot{z}_1$$

$$= \frac{-2\,\eta^3 e}{m(1+\eta^2-\sigma^2)}\left(\frac{E'_-}{\dot{z}_o} - jB'_- + \eta\beta_c r_o B_z\right) \quad , \qquad (12b)$$

$$\left[\left(1+\frac{\sigma^2}{\eta^2}\right)\frac{\partial}{\partial z} + j\beta_e\,\frac{\sigma^2}{\eta^2}\right]\dot{z}_1 - j\,\frac{\sigma\sqrt{1-\eta^2-\sigma^2}}{2\eta^2}\left(\frac{\partial}{\partial z} + j\beta_e\right)(u'_+ - u'_-)$$

$$= -\,\eta\frac{e}{m}\left[\frac{E_z}{\dot{z}_o} - \frac{\eta\,\beta_c r_o}{2}\,(B'_+ + B'_-)\right] \quad . \qquad (12c)$$

The first two of these equations, (12a) and (12b), are analogous to the coupled mode equations for cyclotron waves for a straight filamentary electron beam. The third equation appears here because of the possible a-c motion of the electrons in the axial direction, away from the d-c positions established by the spiral.

Equations analogous to those for the synchronous waves on a straight filamentary electron beam can also be developed. Setting

$$\rho_{\pm}' = \left[u_{\pm} \mp j \frac{2\eta^3\omega_c}{1+\eta^2-\sigma^2} r_{\pm} \right] e^{\mp j\psi} \quad , \tag{13a}$$

$$r_{\pm} = x_1 \pm jy_1 \quad , \tag{13b}$$

$$\tau = \dot{z}_1 - \frac{j\eta^2\beta_e\dot{z}_o}{\eta^2+\sigma^2} z_1 \quad , \tag{13c}$$

the synchronous-type equations are

$$\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_c\right)\left(\rho_+' - \lambda u_-' + j \frac{2\sigma\sqrt{\lambda}}{\sqrt{1+\eta^2-\sigma^2}} \dot{z}_1\right)$$

$$= \frac{-2\eta^3 e}{m(1+\eta^2-\sigma^2)} \left(\frac{E_+'}{\dot{z}_o} + j\mathbf{B}_+' + \eta\beta_c r_o B_z\right) \quad , \tag{14a}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)\left(\rho_-' - \lambda u_+' - j \frac{2\sigma\sqrt{\lambda}}{\sqrt{1+\eta^2-\sigma^2}} \dot{z}_1\right)$$

$$= \frac{-2\eta^3 e}{m(1+\eta^2-\sigma^2)} \left(\frac{E_-'}{\dot{z}_o} - jB_-' + \eta\beta_c r_o B_z\right) \quad , \tag{14b}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)\left[\tau - j\frac{\sigma}{2} \frac{\sqrt{1-\eta^2-\sigma^2}}{\eta^2+\sigma^2} (u_+' - u_-')\right]$$

$$= \frac{-\eta^3 e}{m(\eta^2+\sigma^2)} \left[\frac{E_2}{\dot{z}_o} - \frac{\eta\beta_c r_o}{2} (B_+' + B_-')\right] \tag{14c}$$

Again, the first two equations (14a) and (14b), are directly analogous to the synchronous wave equations in a straight filamentary electron beam, while the third equation appears because of the spiral character of the d-c beam.

D. Coupled Wave Equations

The first goal of this analysis is to develop a set of coupled mode equations which involve the characteristic waves, or eigenfunctions, of the uncoupled beam and circuit systems. For simplicity, this report will consider only circuits with TEM waves which have no variation in a transverse plane. Other circuit wave-types could be considered with no basic change in the method of analysis. The appropriate, circularly polarized, TEM characteristic waves are

$$F'_{\pm} = \frac{1}{4} \sqrt{\frac{A}{Z}} \left(E'_{\pm} \mp j\, Z\, H'_{\pm} \right) \quad , \tag{15a}$$

$$G'_{\pm} = \frac{1}{4} \sqrt{\frac{A}{Z}} \left(E'_{\pm} \pm j\, Z\, H'_{\pm} \right) \quad . \tag{15b}$$

Here A is the cross sectional area of the circuit, and $Z = \sqrt{\mu_o/\epsilon_o}$ is the impedance. The $F'_{\pm}$ are waves traveling in the +z direction, while the $G'_{\pm}$ are waves traveling in the -z direction. The normalization is chosen so that the net time-average power flow in the +z direction is

$$P_{ave} = F_+' F_+'^* + F_-' F_-'^* - G_+' G_+'^* - G_-' G_-'^* \quad . \tag{16}$$

The introduction of Maxwell's equations will give four additional coupled mode equations relating the circuit fields to the electron beam quantities. These are not given here, but appear later in their final form.

The dependent variables for the electron beam quantities which appear in Equations (12) and (14) are not the eigenfunctions for the uncoupled beam in this case where the d-c motion is a spiral. Therefore, more appropriate variables are introduced which will be eigenfunctions for the uncoupled electron beam. These are found to be:

$$P_+' = M\left[u_+' + u_-' + j\,\frac{\eta\beta_c}{\beta_e}\,\frac{4\sigma\sqrt{\lambda}}{\eta^2(1+\lambda)^2}\left[\eta^2(1+\lambda)+\sigma^2(1-\lambda)\right]\frac{\dot{z}_1}{\sqrt{1+\eta^2-\sigma^2}}\right] \tag{17a}$$

$$P_-' = M\left[u_+' - u_-' + j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\,\frac{\dot{z}_1}{\sqrt{1+\eta^2-\sigma^2}}\right], \tag{17b}$$

$$V' = M\,\frac{\dot{z}_1}{\sqrt{1+\eta^2-\sigma^2}}, \tag{17c}$$

$$Q_+' = M\left[\rho_+' - \lambda u_-' + j2\sigma\sqrt{\lambda}\,\frac{\dot{z}_1}{\sqrt{1+\eta^2-\sigma^2}}\right], \tag{17d}$$

$$Q'_- = M \left[\rho'_- - \lambda u'_+ - j2\sigma\sqrt{\lambda}\ \frac{\dot{z}_1}{\sqrt{1+\eta^2-\sigma^2}} \right] , \qquad (17e)$$

$$W' = M \left[\frac{\tau}{\sqrt{1+\eta^2-\sigma^2}} - j\ \frac{\sigma\sqrt{\lambda}}{2(\eta^2+\sigma^2)}\ (u'_+ - u'_-) \right] , \qquad (17f)$$

where the normalizing factor M is

$$M = \frac{1}{4} \sqrt{\frac{1+\eta^2-\sigma^2}{\sqrt{1-\eta^2}}\ \frac{\sigma\ m\ I_o}{\eta^3 e}} . \qquad (18)$$

With these new beam variables, and the introduction of a coupling coefficient K,

$$K = \sqrt{\frac{Z}{A}} \sqrt{\frac{1}{\sqrt{(1-\eta^2)(1+\eta^2-\sigma^2)}}\ \frac{\eta^3 e\ I_o}{\sigma\ c^2\ m}} , \qquad (19)$$

the final coupled mode equations for the interaction between a spiraling filamentary electron beam and a TEM circuit are obtained.

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)P_+' + j\,\frac{2\lambda}{1-\lambda}\,\eta\beta_c P_-' + K\left\{\frac{1-\sigma}{1-\lambda}\,(F_+' + F_-') + \frac{1+\sigma}{1-\lambda}\,(G_+' + G_-')\right.$$

$$\left. + \frac{\sigma\lambda}{\eta^2(1+\lambda)}\;\frac{\eta\beta_c}{\beta_e}\left[\left(1+\eta^2 N - \frac{2\sigma}{1+\lambda}\right)(F_+'-F_-') - \left(1+\eta^2 N + \frac{2\sigma}{1+\lambda}\right)(G_+'-G_-')\right]\right\} = 0 \qquad (20a)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)P_-' + \frac{K}{1+\lambda}\left[(1-\sigma)(F_+'-F_-') + (1+\sigma)(G_+'-G_-')\right] = 0 \qquad (20b)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\,\frac{\sigma^2(1-\lambda)}{\eta^2(1+\lambda)N}\right)V' - j\,\frac{\sqrt{\lambda}}{4}\,\frac{(1+\eta^2-\sigma^2)}{\eta^2 N}\,K\left\{\left[1 - \frac{2\sigma(1-\sigma)}{(1+\eta^2-\sigma^2)(1+\lambda)}\right](F_+'-F_-')\right.$$

$$\left. - \left[1 + \frac{2\sigma(1+\sigma)}{(1+\eta^2-\sigma^2)(1+\lambda)}\right](G_+' - G_-')\right\} = 0 \qquad (20c)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_c\right)Q_+' + K\left[(1-\sigma)F_+' + (1+\sigma)G_+'\right] = 0 \qquad (20d)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)Q_-' + K\left[(1-\sigma)F_-' + (1+\sigma)G_-'\right] = 0 \qquad (20e)$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)W' - j\,\frac{\sqrt{\lambda}}{4}\,\frac{1+\eta^2-\sigma^2}{\eta^2+\sigma^2}\,K\left[F_+' - F_-' - G_+' + G_-'\right] = 0 \tag{20f}$$

$$\left(\frac{\partial}{\partial z} + j\sigma\beta_e + j\eta\beta_c\right)F_+' - \frac{K}{2}\left[P_+' + P_-' - j\,\frac{4\,\sigma\sqrt{\lambda}}{1+\lambda}\left(1 + \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{20g}$$

$$\left(\frac{\partial}{\partial z} + j\sigma\beta_e - j\eta\beta_c\right)F_-' - \frac{K}{2}\left[P_+' - P_-' + j\,\frac{4\sigma\sqrt{\lambda}}{1+\lambda}\left(1 - \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{20h}$$

$$\left(\frac{\partial}{\partial z} - j\sigma\beta_e + j\eta\beta_c\right)G_+'\,\frac{K}{2}\left[P_+' + P_-' - j\,\frac{4\,\sigma\sqrt{\lambda}}{1+\lambda}\left(1 + \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{20i}$$

$$\left(\frac{\partial}{\partial z} - j\sigma\beta_e - j\eta\beta_c\right)G_-' + \frac{K}{2}\left[P_+' - P_-' + j\,\frac{4\,\sigma\sqrt{\lambda}}{1+\lambda}\left(1 - \frac{\eta\beta_c}{\beta_e}N\right)V'\right] = 0 \tag{20j}$$

$$N = \frac{\eta^2(1+\lambda) + \sigma^2(1-\lambda)}{\eta^2\,(1+\lambda)} \tag{21}$$

These ten coupled wave equations include six beam waves, P_{+}', P_{-}', V', Q_{+}', Q_{-}', W', and four circuit waves, F_{+}', F_{-}', G_{+}', and G_{-}'. The first three equations, (20a, b, c), involve the coupling between the cyclotron-type waves on the beam and the circuit waves. These equations are obtained by taking appropriate combinations of Equations (12a, b, c). The next three equations (20d, e, f), involve the coupling between the synchronous-type waves on the beam and the circuit waves; these follow directly from Equations (14a, b, c). The final four equations (20g, h, i, j), involve the coupling between the TEM circuit waves and the cyclotron-type waves on the beam. These equations are just Maxwell's equations rewritten in the form appropriate to this coupled mode analysis.

E. Discussion of the Waves

Prior to a consideration of the possible interactions between a spiraling filamentary electron beam and a TEM circuit, several comments concerning the characteristic waves of the uncoupled beam should be made. These characteristic waves, or eigenfunctions, of the electron beam (and of the circuit) are obtained by setting the coupling coefficient K equal to zero in Equations (20) and solving them. Recalling that exp ($j\omega t$) was understood for the a-c quantities, the ten uncoupled waves are readily found.

$$P'_+ = P'_{+o}\, e^{j\omega t - j\beta_e z} - j\,\frac{2\lambda\eta\beta_c}{1-\lambda}\, z\, P'_- \tag{22a}$$

$$P'_- = P'_{-o}\, e^{j\omega t - j\beta_e z} \tag{22b}$$

$$V' = V'_0\, e^{j\omega t - j\beta_e \frac{\sigma^2(1-\lambda)}{\eta^2(1+\lambda)N} z} \tag{22c}$$

$$Q'_+ = Q'_{+o}\, e^{j\omega t - j(\beta_e + \eta\beta_c)z} \tag{22d}$$

$$Q'_- = Q'_{-o}\, e^{j\omega t - j(\beta_e - \eta\beta_c)z} \tag{22e}$$

$$W' = W'_o\, e^{j\omega t - j\beta_e z} \tag{22f}$$

$$F'_+ = F'_{+o}\, e^{j\omega t - j(\sigma\beta_e + \eta\beta_c)z} \tag{22g}$$

$$F'_- = F'_{-o}\, e^{j\omega t - j(\sigma\beta_e - \eta\beta_c)z} \tag{22h}$$

$$G'_+ = G'_{+o}\, e^{j\omega t + j(\sigma\beta_e - \eta\beta_c)z} \tag{22i}$$

$$G'_- = G'_{-o}\, e^{j\omega t + j(\sigma\beta_e + \eta\beta_c)z} \tag{22j}$$

The ω-β diagrams for these uncoupled waves are shown in Figure 3; the six electron beam waves are given by the solid lines, and the four circuit waves by the dashed lines. Note that the inclusion of the phase factor Ψ in the definitions of these waves has produced a shift in the intersection point on the β axis for many of them (compared to the corresponding ω-β diagrams for a straight filamentary electron beam, for example). The ten waves can be divided into three groups on the basis of the magnitudes of their group velocities; i.e., the magnitudes of the slopes of the lines in Figure 3. The magnitudes of the group velocities for the four circuit waves, F'_+, F'_-, G'_+, G'_- are all equal to the velocity of light c. The magnitudes of the group velocities of five of the beam waves, P'_+, P'_-, Q'_+, Q'_-, and W', are all equal to the d-c velocity of the electrons in the axial direction, $\dot{z}_o$. The group velocity of the remaining beam wave, V', will in most cases be greater than c.

Examination of the characteristic waves for the uncoupled beam and circuit system shows that all of them are eigenfunctions except for P'_-. That is, each of the other waves can exist independently of any of the remaining

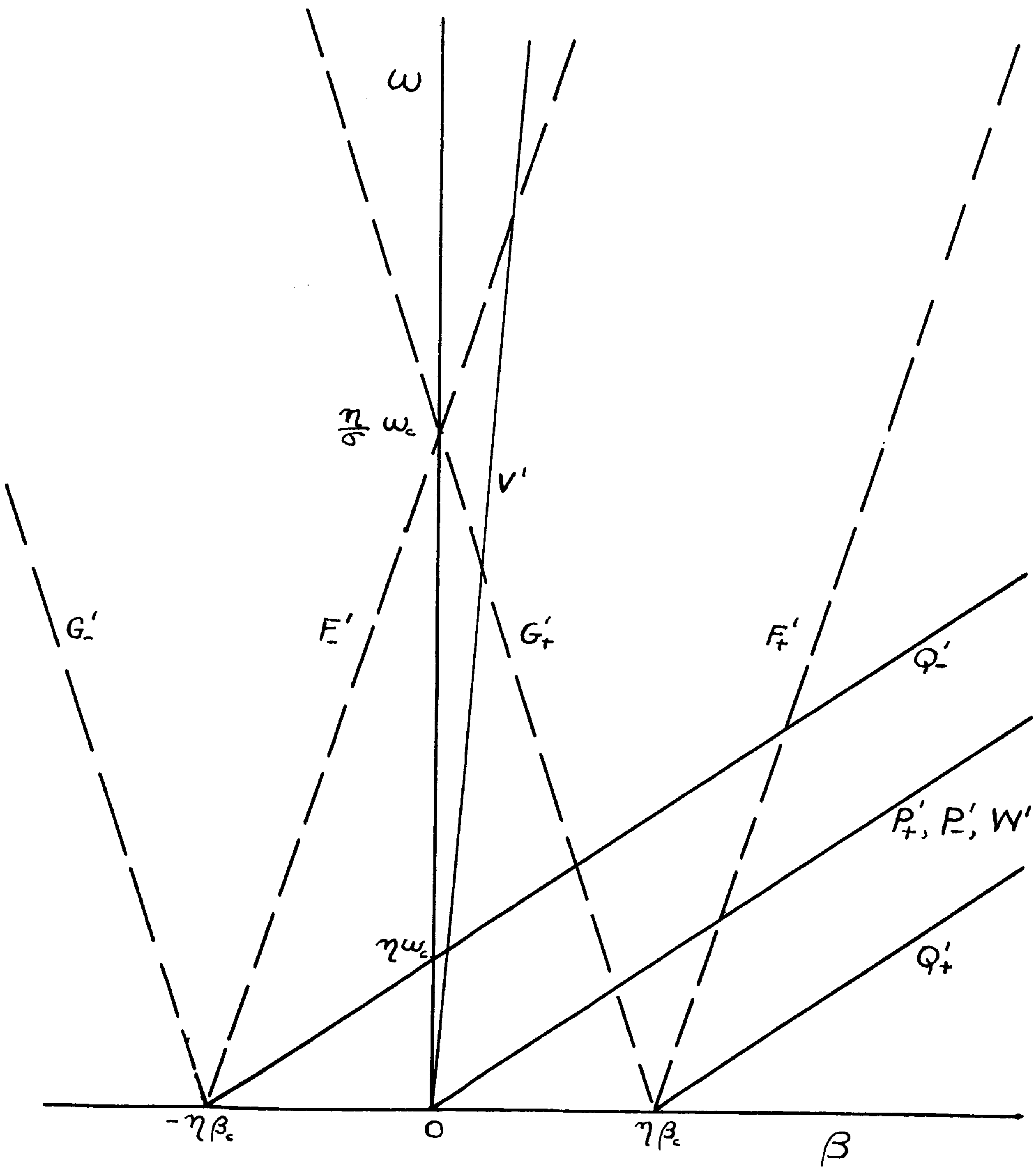

FIGURE 3. The ω-β Diagram for Spiraling Filamentary Electron Beam Waves (Solid Lines) and TEM Circuit Waves (Dashed Lines).

waves. This is not true for P'_-. Whenever P'_- exists, P'_+ will automatically be excited, and moreover, grow linearly with distance axially. Physically this means that the cyclotron-type waves for the uncoupled spiraling filamentary electron beam exhibit a weak instability. Any excitation of the electron beam which couples to the P'_- wave, whether it is a desired signal, or thermal, or other, noise, will produce a P'_+ wave which grows linearly with axial distance.

As in all coupled mode analyses, the possibility for strong coupling between the beam and circuit waves occurrs in the neighborhood of those frequencies for which the phase constants of the waves match; that is, in the neighborhood of the crossing points of the ω-β lines shown in Figure 3. Note that the beam waves (with the exception of P'_+ and P'_- discussed above) will not interact with each other, nor will the circuit waves interact with each other. The only possibilities for strong interaction occur where a solid line intersects a dashed line in Figure 3. However, not all intersections of this type will lead to interactions, because the symmetry of the waves will exclude some possible interactions. For example, a purely positive circularly polarized wave will not interact with a purely negative circularly polarized wave. Considerations of this type rule out strong interaction between the Q'_- wave and either the F'_+ or the G'_+ waves.

Possible strong coupling might occur, then, between the V' wave and either the F'_- or the G'_+ waves; and between a combination of the P'_+ and P'_- waves and either the F'_+ or the G'_+ waves. Interactions involving a combination of the P'_+ and P'_- waves would appear to offer some promising possibilities, based on analogy with the waves on a straight filamentary electron beam.

In the straight filamentary electron beam, the fast cyclotron wave is a positive energy wave, while the slow cyclotron wave is a negative energy wave. This means that when a circuit with a positive energy wave is coupled to the electron beam under conditions for strong interaction between the circuit wave and the slow cyclotron wave, the system may exhibit a net gain and be capable of amplification or oscillation. In the case of the spiraling filamentary electron beam, the characteristic beam waves P'_+ and P'_- each involve both of the cyclotron-type wave components u'_+ and u'_- (see Equations (17a,b)). In a sense, then, the d-c spiraling of the electron beam couples the fast and slow cyclotron waves of the straight electron beam together. This coupling of a negative energy beam wave to a positive energy beam wave explains physically the weak instability of the cyclotron-type beam waves for the spiraling electron beam. It is possible that this coupling may also lead to signal amplification when the spiraling filamentary electron beam interacts with a circuit with appropriate parameters.

This analysis has explored the characteristic waves for a spiraling filamentary electron beam. The next phase of this study should include an investigation of the sign of the power flow associated with the various waves, and equally important, the possible regions of strong interaction between the electron beam waves and the circuit waves to determine which interactions may lead to useful amplifying or oscillating systems. Strong interaction between an electron beam wave and a circuit wave is not a sufficient condition for amplification or oscillation, as the short discussion of the straight filamentary electron beam case of the next section indicates.

F. Straight Filamentary Electron Beam

The theory developed above for the spiraling filamentary electron beam contains the straight filamentary electron beam as a limiting case; where $\lambda = 0$. In this limiting case, however, certain simplifications are possible. First, it is no longer desirable to include the phase factor ψ in the definitions of the characteristic wave variables, and second, the cyclotron waves become uncoupled.

Therefore, more appropriate variables are defined as

$$C_+ = M\, u_+ \quad , \tag{23a}$$

$$C_- = M\, u_- \quad , \tag{23b}$$

$$S_+ = M\, \rho_+ \quad , \tag{23c}$$

$$S_- = M\, \rho_- \quad , \tag{23d}$$

where M is as given in Equation (18) with $\lambda = 0$ (this implies $\eta^2 = 1 - \sigma^2$). The TEM circuit waves are defined as before, but the phase factor ψ is dropped, leading to the unprimed variables F_+, F_-, G_+, G_-. The coupled wave equations, (20a-j) now become

$$\left(\frac{\partial}{\partial z} + j\beta_e - j\eta\beta_c\right)C_+ + K\left[(1-\sigma)F_+ + (1+\sigma)G_+\right] = 0 \ , \tag{24a}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e + j\eta\beta_c\right)C_- + K\left[(1-\sigma)F_- + (1+\sigma)F_-\right] = 0 \ , \tag{24b}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)S_+ + K\left[(1-\sigma)F_+ + (1+\sigma)G_+\right] = 0 \quad , \tag{24c}$$

$$\left(\frac{\partial}{\partial z} + j\beta_e\right)S_- + K\left[(1-\sigma)F_- + (1+\sigma)G_-\right] = 0 \quad , \tag{24d}$$

$$\left(\frac{\partial}{\partial z} + j\sigma\beta_e\right)F_+ - KC_+ = 0 \quad , \tag{24e}$$

$$\left(\frac{\partial}{\partial z} + j\sigma\beta_e\right)F_- - KC_- = 0, \tag{24f}$$

$$\left(\frac{\partial}{\partial z} - j\sigma\beta_e\right)G_+ + KC_+ = 0 \quad , \tag{24g}$$

$$\left(\frac{\partial}{\partial z} - j\sigma\beta_e\right)G_- + KC_- = 0 \quad . \tag{24b}$$

The C_+ and C_- waves are the fast and slow cyclotron waves, while the S_+ and S_- are the synchronous waves.

The ω-β diagrams for the uncoupled electron beam and circuit waves ($K = 0$) are shown in Figure 4; again the beam waves are solid and circuit waves are dashed. There is the possibility of strong coupling where the lines cross, that is, between C_+ and F_+ or G_+ (C_+ will not couple to F_- or G_- because the sense of circular polarization of the beam wave does not match that of the circuit waves). Figure 5 shows the resulting diagram for the C_+, S_+, F_+, and G_+ waves when the coupling is accounted for in a typical case (the dotted curves indicate the imaginary part of a complex phase constant).

These results show that strong interaction does, indeed, occur between the C_+ and G_+ waves in the general vicinity of $\omega = \eta\omega_c/(1-\sigma)$, and between the C_+ and F_+ waves in the general vicinity of $\omega = \eta\omega_c/(1+\sigma)$. In fact, the interaction between the C_+ and G_+ waves leads to a pair of complex conjugate values for the phase constant β. However, an examination of the various configurations of a straight filamentary electron beam with a TEM circuit

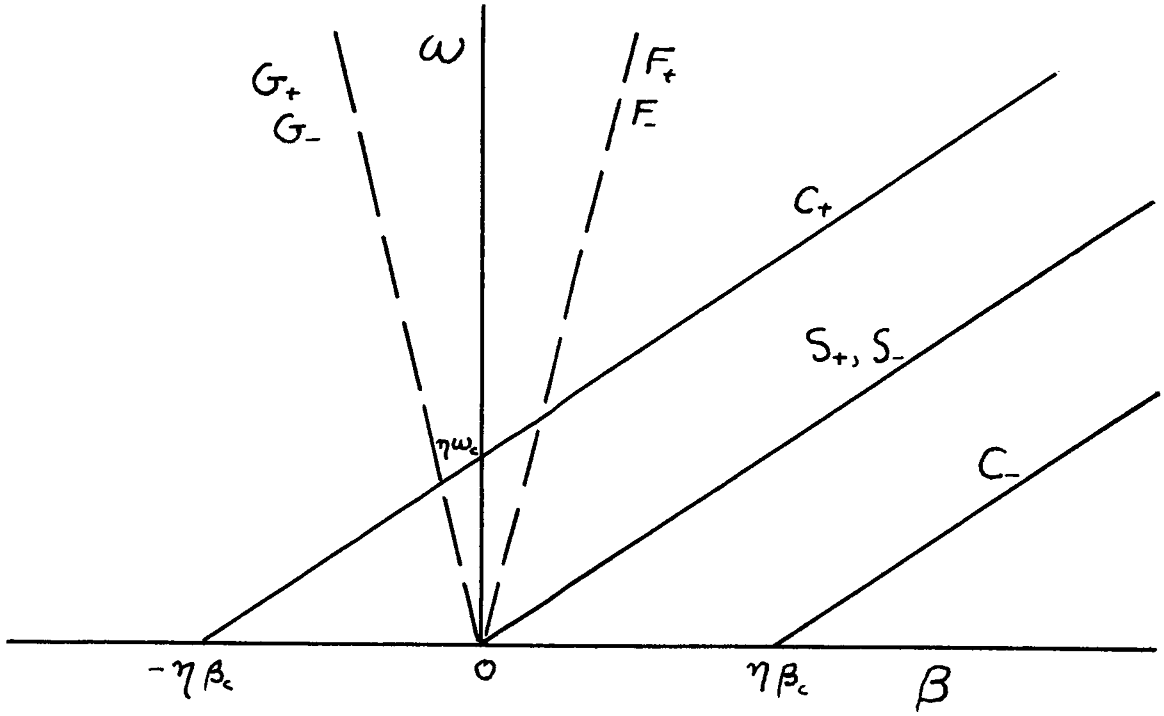

FIGURE 4. The ω-β Diagram for Straight Filamentary Electron Beam Waves (Solid Lines) and TEM Circuit Waves (Dashed Lines).

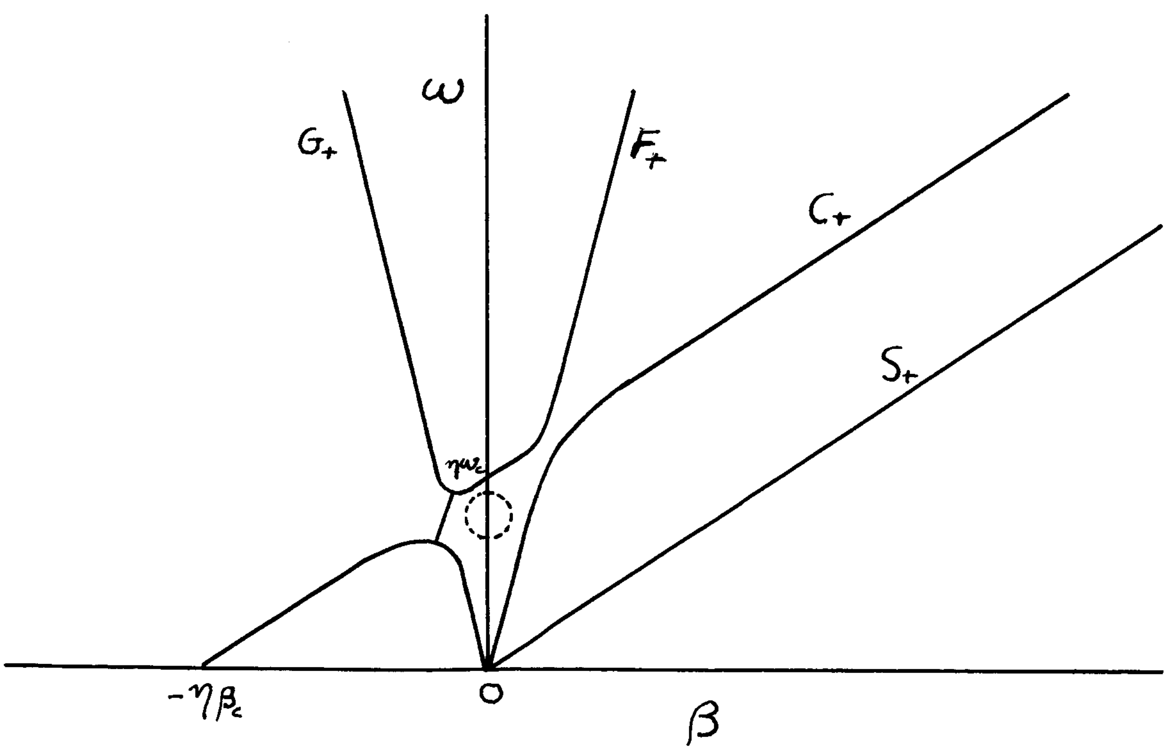

FIGURE 5. The ω-β Diagram for Coupled Waves of a Straight Filamentary Electron Beam and a TEM Circuit(Complex Values for β- Dotted Lines).

shows that it is not possible to attain amplification or oscillation with this system. Physically the reason is that the waves involved are all positive energy waves, and it is not possible to extract more a-c energy than is supplied to the system by coupling these waves together.

III. RELATIVISTIC SOLID ELECTRON BEAMS

The study of magnetically focused, solid electron beams including relativistic effects has continued. The analysis of the d-c characteristics of a Brillouin beam and a uniform charge density beam were discussed in the previous semiannual status report.[7] In the interim, work on the d-c uniform charge density beam has been completed. In addition, a study of some of the modes of a relativistic Brillouin beam has been undertaken to explore the possibilities for interaction between this beam and a uniform circuit. This research complements that discussed above for the filamentary electron beam which neglects space charge effects. Here, at the expense of considerably more complexity, space charge effects are included.

A. Uniform Charge Density Beam

The uniform charge density beam is one in which the electron charge density is uniform with radius. In addition, for simplicity it is also assumed that the total axial magnetic flux density (applied plus self magnetic fields) is independent of radius. The basic equations for this electron beam were presented in the previous report[7] and will not repeated here. A digital computer has been employed to obtain several of the more important beam quantities as a function of the radius and the beam voltage. These are presented in the following figures.

Figures 6 and 7 show the normalized angular and axial velocities for this beam as a function of the normalized radius within the beam. Similar curves were presented in the previous report[7], but for a much more restricted range of radii. As for the parameters which appear, it is recalled that Ω is related to the axial velocity of the beam at its center,

$$\Omega = \left(1 - \frac{\dot{z}^2(o)}{c^2}\right)^{-1/2} , \tag{25}$$

ω_c is the cyclotron frequency associated with the total magnetic flux density (this is independent of radius), and η is a measure of the total magnetic flux linking the cathode. For $\eta = 1.0$, no magnetic flux links the cathode; for $\eta = 0.5$, half the magnetic flux links the cathode (in the same direction as in the drift region); and for $\eta = 1.5$, half the magnetic flux links the cathode but in a direction opposite to that in the drift region.

Figure 8 shows the required radial distribution of the applied axial magnetic flux density in the drift region to focus the electron beam. Here ω_a is the cyclotron

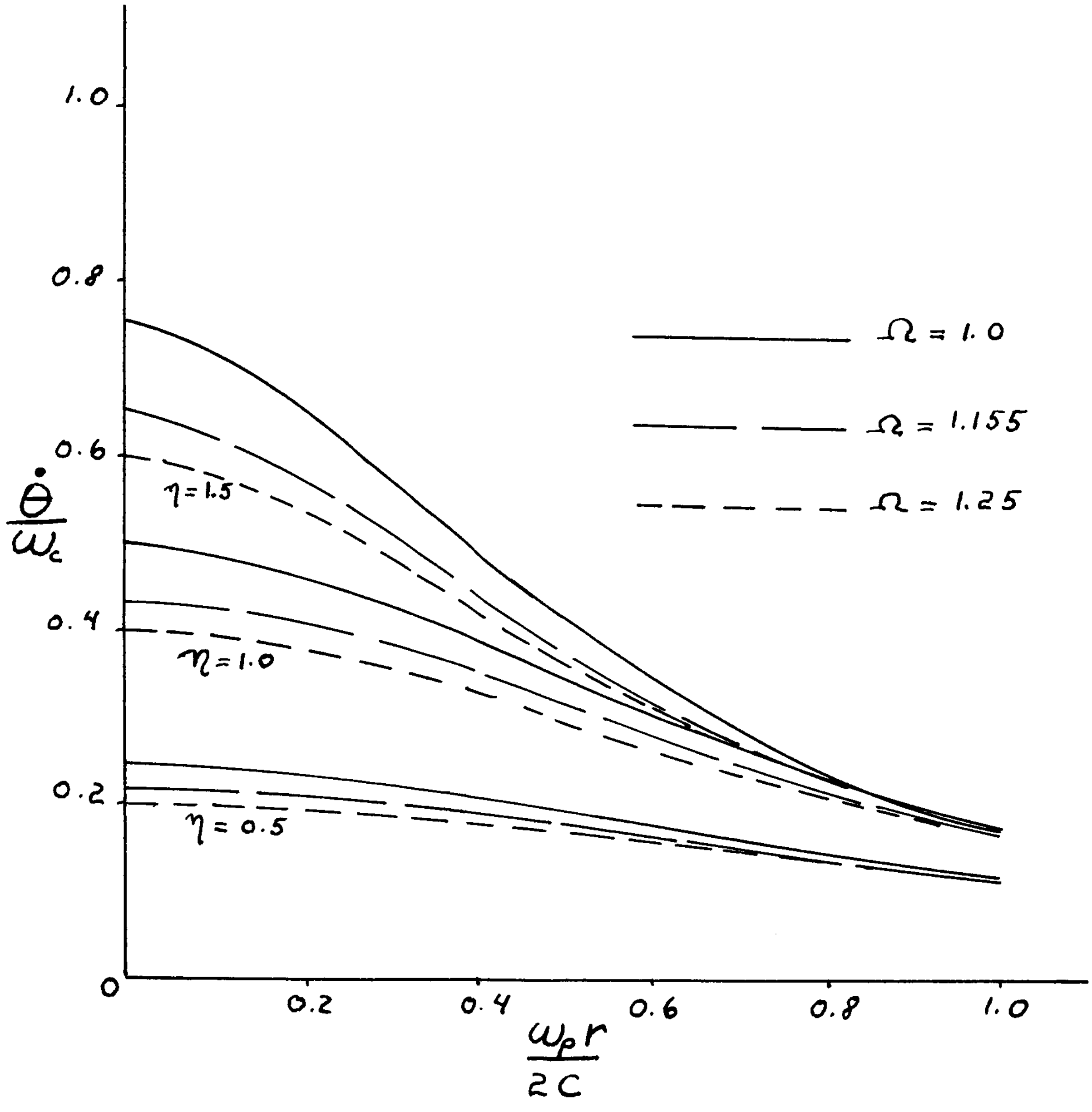

FIGURE 6. Normalized Angular Velocity $\frac{\dot{\theta}}{\omega_c}$ Versus Normalized Radial Position $\frac{\omega_p r}{2c}$ for an Uniform Charge Density Beam.

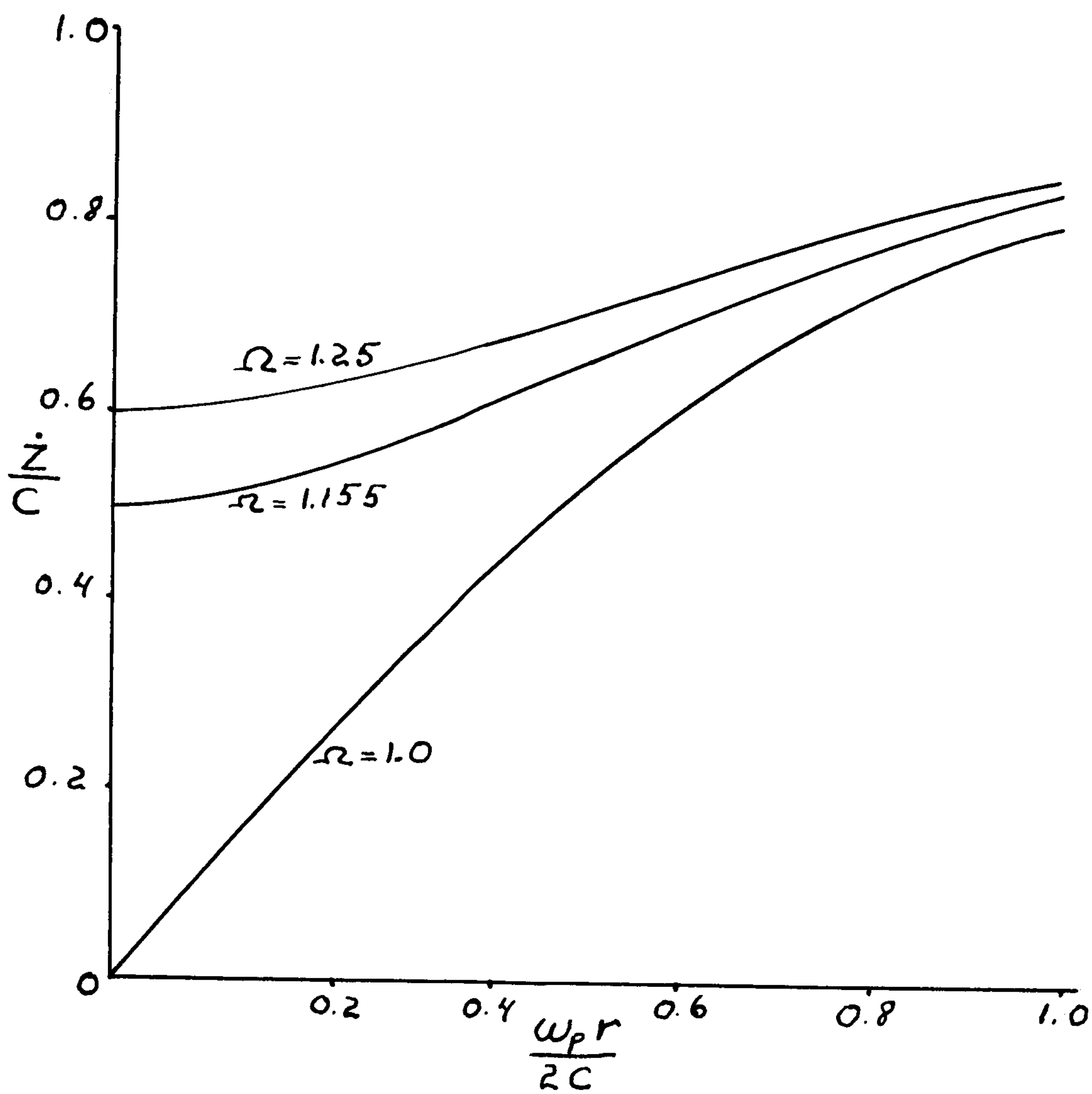

FIGURE 7. Normalized Axial Velocity $\frac{\dot{z}}{c}$ Versus Normalized Radial Position $\frac{\omega_p r}{2c}$ for an Uniform Charge Density Beam.

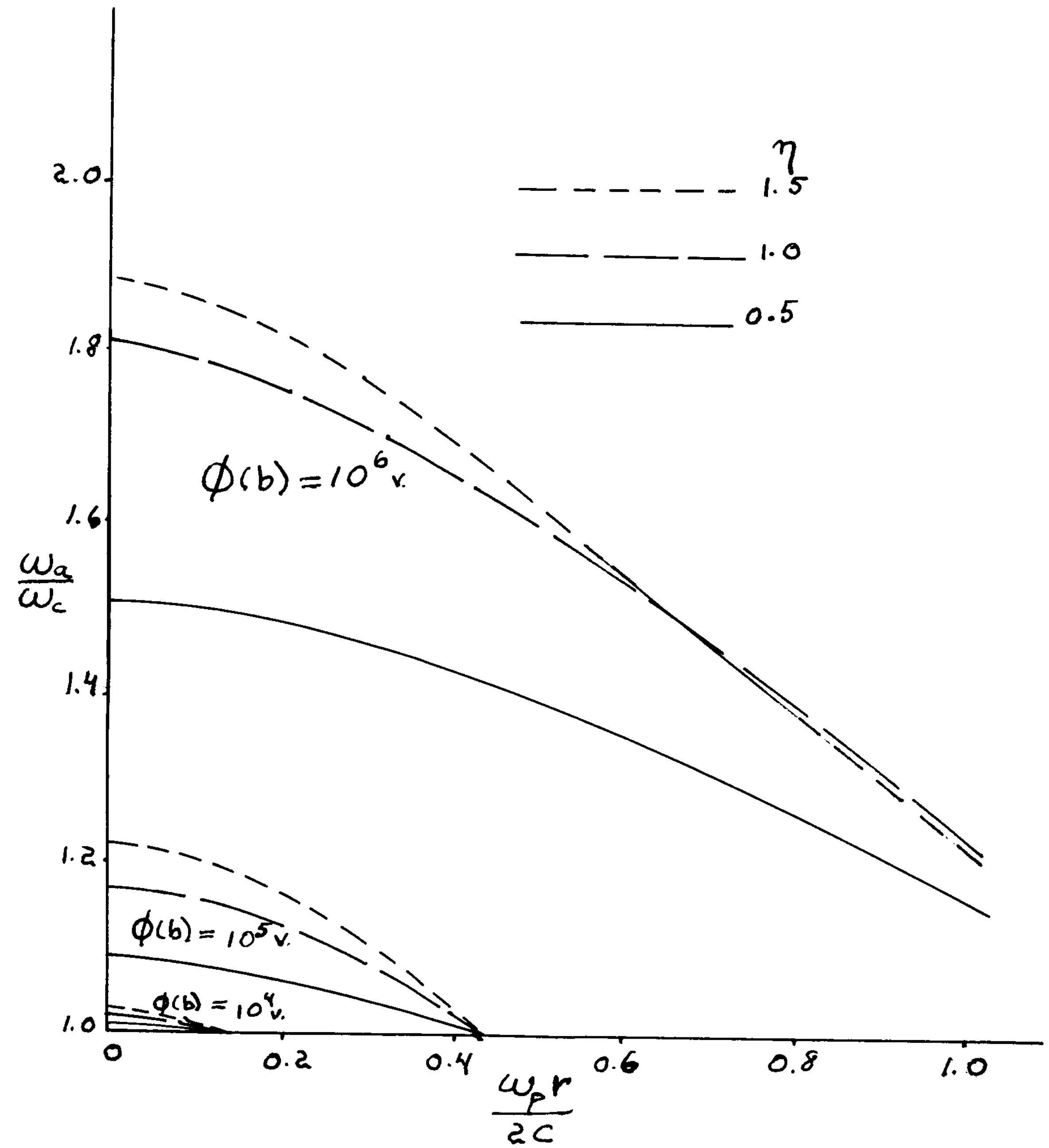

FIGURE 8. Normalized Applied Magnetic Flux Density ω_a/ω_c Versus Normalized Radial Position $\frac{\omega_p r}{2c}$ for an Uniform Charge Density Beam.

frequency for the applied magnetic flux density. The parameters are η and the total equivalent d-c beam voltage at the beam edge, $\phi(b)$.

Finally, Figure 9 shows the d-c beam conductance, $I/\phi(b)$, versus Ω for several values of the d-c beam voltage.

B. Modes of a Brillouin Beam

With the d-c state of several representative relativistic solid electron beams determined, an analysis of the possible modes of these beams has been initiated. The Brillouin beam has been selected as the first case to be studied, because it is believed that the analysis of its modes will be simpler than for any other d-c beam; therefore, it is an appropriate introduction to this type of study.

The analysis follows, in general, the development of previous studies of the modes of magnetically focused electron beams.[8-10] The main difference is that here relativistic effects are included, and attention is centered on the transverse modes which are analogous to the cyclotron waves of a filamentary beam. For this analysis, four simplifying approximations have been made: (1) relativistic effects have been included by retaining

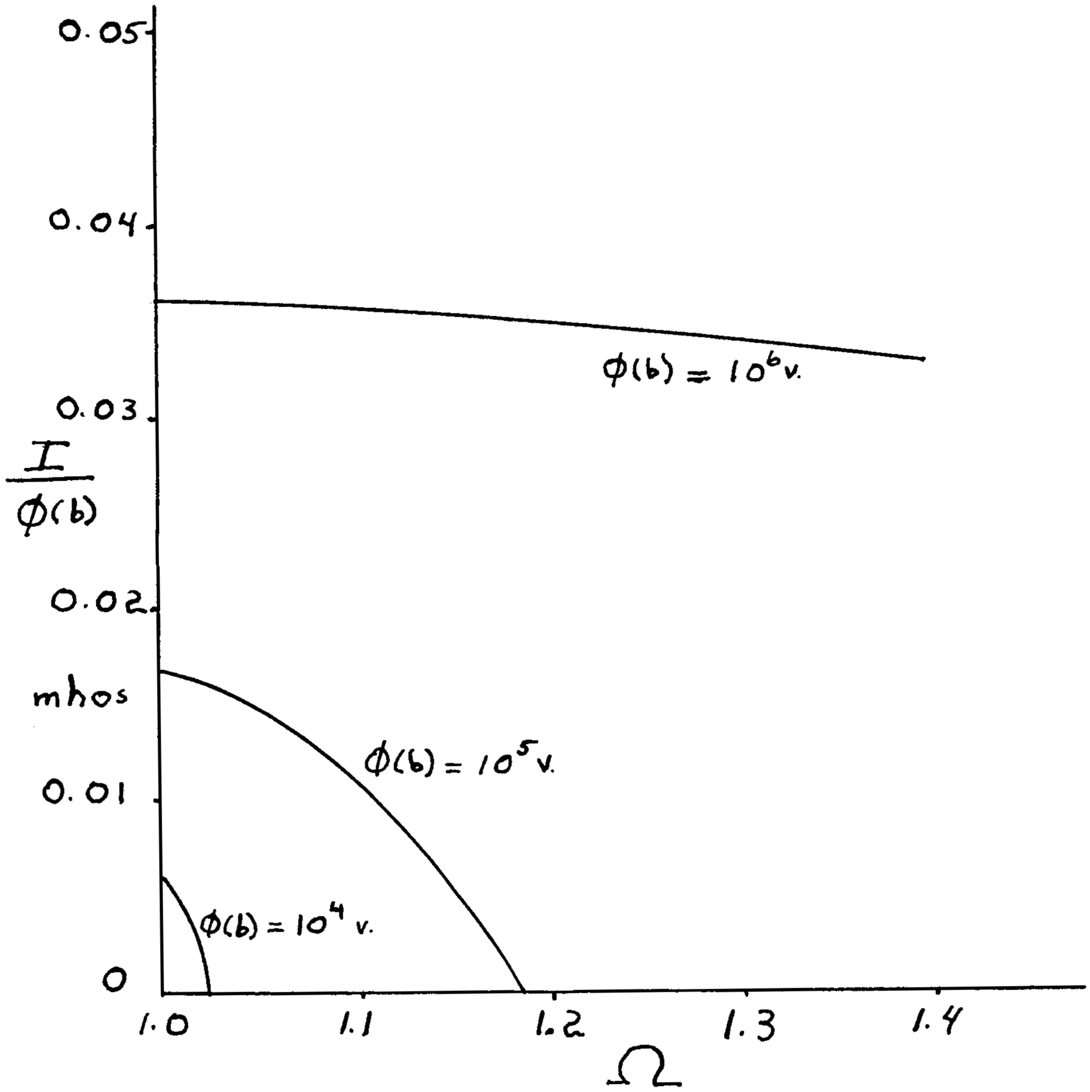

FIGURE 9. Electron Beam Conductance $I/\phi(b)$ Versus Ω for an Uniform Charge Density Beam.

terms up to order $(v/c)^2$; (2) the beam is thin, i.e., the radius of the beam is small compared to the wavelength; (3) only fast waves on the beam are considered (since possible interaction with uniform circuits is of primary interest); and (4) only waves with azimuthal variations are considered (ordinary space charge waves are not included).

The general procedure is to combine the relativistic equations of motion (linearized for a small signal analysis) with Maxwell's equations. The d-c quantities to be used for the beam are those for a Brillouin beam[7]. By appropriate manipulations, all the a-c beam quantities and the electromagnetic fields can be expressed in terms of the two a-c electric field components, $E_{1\theta}$, and E_{1z}, and the d-c beam parameters. Coupled partial differential equations for these two electric field components have been derived.

$$\frac{\partial^2 E_{1\theta}}{\partial r^2} + \frac{3}{r}\frac{\partial E_{1\theta}}{\partial r} - \frac{n^2-1}{r^2} E_{1\theta} - \left(\frac{n\gamma\ p^2 c^2}{2\omega^2 r\dot{\theta}_o^2} - \frac{n\dot{z}_o}{\omega r}\right)\left[\frac{\partial^2 E_{1z}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{1z}}{\partial r} - \frac{n^2}{r^2} E_{1z}\right] = 0, \tag{26a}$$

$$\frac{\partial^2 E_{1z}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{1z}}{\partial r} - \frac{n^2}{r^2} E_{1z} - \left(\frac{\gamma r}{n} - \frac{2\omega r\dot{\theta}_o^2 \dot{z}_o}{np^2c^2}\right)\left[\frac{\partial^2 E_{1\theta}}{\partial r^2} + \frac{3}{r}\frac{\partial E_{1\theta}}{\partial r} - \frac{(n^2-1)}{r^2} E_{1\theta}\right] = 0 \quad . \tag{26b}$$

Here γ is the axial phase constant, n is an integer specifying the azimuthal variation, and $p = (\omega - n\dot{\theta}_o - \gamma\dot{z}_o)$. These equations must now be solved in conjunction with appropriate boundary conditions at the beam edge to determine the modes of the beam.

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